Lie Theorem for superposition rules for PDEs revisited

2$^{\text{nd}}$ International Workshop on Geometric Field Theory

7$^{\text{th}}$ July 2026

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Joint work with J. de Lucas

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Outline

Let $J^r\pi$ be the r^{th}-jet space of a fibre bundle $\pi\colon E \to M$.

Under which conditions the general solution of a PDE

$$\mathcal{E} \subset J^r\pi$$

can be expressed in terms of a finite number of particular solutions?

This is the case of two relevant families:

- Darboux integrable PDEs (Anderson, Fels, Vassiliou, Ivey):

$$u_{xy} = \mathrm{e}^u \rightsquigarrow u(x,y) := \log \frac{2\dot{f}(x)\dot{g}(y)}{(f(x)+g(y))^2}.$$

- **Lie systems** (Cariñena, Grabowski, Marmo, Winternitz, Odzijewicz, Grundland, de Lucas, Sardón):

$$\dot{x} = b_0(t) + b_1(t)x + b_2(t)x^2 \rightsquigarrow x(t) := \frac{x_{(1)}(x_{(3)}-x_{(2)}) - kx_{(2)}(x_{(3)}-x_{(1)})}{(x_{(3)}-x_{(2)}) - k(x_{(3)}-x_{(1)})}.$$

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First-order PDEs

- $\pi : \mathbb{R}^s \times M \longrightarrow \mathbb{R}^s$ with adapted coordinates (t^i, u^α).
- $J^1\pi$ with adapted coordinates $(t^i, u^\alpha, u^\alpha_i)$.

Let now $\mathbf{X}$ be a section of $\pi_{1,0} : J^1\pi \longrightarrow \mathbb{R}^s \times M$:

$$\begin{tikzcd} & J^1\pi \arrow[dl, "\pi_{1,0}"] \arrow[dd] \\ \mathbb{R}^s \times M \arrow[ur, bend left, "\mathbf{X}"] \arrow[dr, "\pi"'] & \\ & \mathbb{R}^s \end{tikzcd}$$

$\mathcal{E} := \mathbf{X}(M) \subset J^1\pi$ is a **PDE on M in normal form**

A **(local) solution** of $\mathcal{E}$ is a local section $\phi : \mathbb{R}^s \dashrightarrow \mathbb{R}^s \times M$ of $\pi : \mathbb{R}^s \times M \longrightarrow \mathbb{R}^s$ such that $\mathrm{Im}(j^1\phi) \subset \mathcal{E}$.

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s-vector fields

- An **s-vector field** is a section of the Whitney sum $\oplus^s \mathrm{T}M \to M$.
- A **t-dependent s-vector field**, with $t = (t^1, \dots, t^s) \in \mathbb{R}^s$, is a section $\mathbf{X}$ of the Whitney sum

$$\oplus^s \mathrm{pr}_M^* \mathrm{T}M \longrightarrow \mathbb{R}^s \times M, \qquad (\mathrm{pr}_M \colon \mathbb{R}^s \times M \to M).$$

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- Associated system

$$\frac{\partial u}{\partial t^i} = X_i(t, u).$$

A particular solution $\gamma \colon \mathbb{R}^s \dashrightarrow M$ with $\gamma(0) = u_0$ is called an **integral section** of $\mathbf{X}$ with initial condition u_0 at $t = 0$.

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$$\{\text{germs of integral sections of } \mathbf{X}\} \quad \leftrightarrow \quad \{\text{solutions of } \mathcal{E}\}$$

- $\mathbf{X} = (X_1, \dots, X_s)$ is **integrable** if the autonomisations

$$\widetilde{X}_i := \frac{\partial}{\partial t^i} + X_i(t, u) \in \mathfrak{X}(\mathbb{R}^s \times M)$$

satisfy $[\widetilde{X}_i, \widetilde{X}_j] = 0$.
- Equivalently for the corresponding $\mathcal{E} \subset J^1\pi$:

$$\mathcal{E} \ni z \mapsto (C^1_\pi)_z \cap \mathrm{T}_z\mathcal{E} \subset \mathrm{T}_z\mathcal{E}$$

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Superposition rules

Let us consider the **Riccati PDE system** on $\mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t^1} = b_0^1(t) + b_1^1(t)u + b_2^1(t)u^2, \\ \dfrac{\partial u}{\partial t^2} = b_0^2(t) + b_1^2(t)u + b_2^2(t)u^2. \end{cases}$$

Locally defined map $\Phi\colon \mathbb{R}^3 \times \mathbb{R} \dashrightarrow \mathbb{R}$

$$\Phi(u_{(1)}, u_{(2)}, u_{(3)}; k) := \frac{u_{(1)}(u_{(3)} - u_{(2)}) - ku_{(2)}(u_{(3)} - u_{(1)})}{(u_{(3)} - u_{(2)}) - k(u_{(3)} - u_{(1)})}.$$

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$$(u_{(1)}(0), u_{(2)}(0), u_{(3)}(0)) \in U := \left\{u_{(i)} \neq u_{(j)}\right\} \subset \mathbb{R}^3,$$

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Definition (**Superposition rule**)

A **superposition rule** for $\mathbf{X}$ depending on $\ell \geq 1$ particular solutions is

$$\Phi\colon M^\ell \times M \dashrightarrow M$$

satisfying

(i) $\Phi_p := \Phi(p; \cdot)\colon M \dashrightarrow M$ is étale for every $p \in M^\ell$ where it is defined and,

(ii) there exist open subsets $U \subset M^\ell$ and $W \subset M$ such that, given particular solutions $u_{(1)}(t), \ldots, u_{(\ell)}(t)$ with $(u_{(1)}(0), \ldots, u_{(\ell)}(0)) \in U$, every particular solution $u(t)$ with $u(0) \in W$ can be cast in the form

$$u(t) = \Phi(u_{(1)}(t), \ldots, u_{(\ell)}(t); k),$$

where $k \in M$ is related to the initial conditions.

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Definition (**PDE Lie system**)

$\mathbf{X} = (X_1, \ldots, X_s)$ is a **PDE Lie system** if there exists a finite-dimensional Lie algebra of vector fields $V \subset \mathfrak{X}(M)$ (a so-called **VG Lie algebra**) such that $(X_i)_t \in V$ for every $t \in \mathbb{R}^s$ and $1 \leq i \leq s$.

Theorem (Lie; Odzijewicz–Grundland; Cariñena–Grabowski–Marmo)

$\mathbf{X}$ *admits a superposition rule if and only if it is a PDE Lie system.*

Riccati PDE system

$\mathbf{X} = (X_1, X_2)$, where

$$X_i := \left(b_0^i(t) + b_1^i(t)u + b_2^i(t)u^2\right)\partial_u$$

take values in

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Outline

Diagonal prolongations

- The ℓ^{th}**-diagonal prolongation** of a VB $\tau_E \colon E \longrightarrow M$ is the VB over M^ℓ with:
  - total space $E^{[\ell]} := E \times \overset{\ell}{\cdots} \times E$;
  - projection $\tau_E^{[\ell]} := \tau_E \times \overset{\ell}{\cdots} \times \tau_E$, and
  - the fibre-wise vector space structure on $E^{[\ell]}_{(u_{(1)},\dots,u_{(\ell)})} = E_{u_{(1)}} \times \overset{\ell}{\cdots} \times E_{u_{(\ell)}}$.
- It can be identified with the Whitney sum

$$E^{[\ell]} = \mathrm{pr}_1^* E \oplus \cdots \oplus \mathrm{pr}_\ell^* E \rightsquigarrow (\mathrm{T}M)^{[\ell]} = \mathrm{T}M^\ell$$

$$\Gamma(E) \ni e \mapsto e^{[\ell]} \in \Gamma\left(E^{[\ell]}\right)$$

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- Given a VG Lie algebra $V \subset \mathfrak{X}(M)$, there exists $\ell \geq 1$ such that

$$V^{[\ell]} := \langle Y^{[\ell]} : Y \in V \rangle \simeq V$$

possesses a modular basis; namely, a basis $\left\{X_1^{[\ell]}, \ldots, X_r^{[\ell]}\right\}$ such that

$$X_1^{[\ell]} \wedge \cdots \wedge X_r^{[\ell]}$$

is nowhere vanishing on an open and dense subset $U \subset M^{\ell}$.

Let $\mathbf{X} = (X_1, \dots, X_s)$ be a PDE Lie system on M.

Definition (Trivialising submersion)

A **trivialising submersion** for $\mathbf{X}$ is a submersion $\Psi \colon M \times M^\ell \dashrightarrow M$ such that

(i) $\mathrm{T}\Psi\left((X_i)_t^{[\ell+1]}|_{\mathrm{Dom}(\Psi)}\right) = 0$ for all $t \in \mathbb{R}^s$, $1 \leq i \leq s$, and

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$\mathbf{X}$ *admits a trivialising submersion* $\Psi \colon M \times M^\ell \dashrightarrow M$ *if and only if there exists a VG Lie algebra* $V \subset \mathfrak{X}(M)$ *for* $\mathbf{X}$ *such that* $V^{[\ell]}$ *possesses a modular basis.*

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Theorem

Given $(p_0, k_0) \in M^\ell \times M$ and $u_0 \in M$, there is a 1-1 correspondence

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$\ell = 3$ and $\Psi\colon \mathbb{R} \times \mathbb{R}^3 \dashrightarrow \mathbb{R}$ given by

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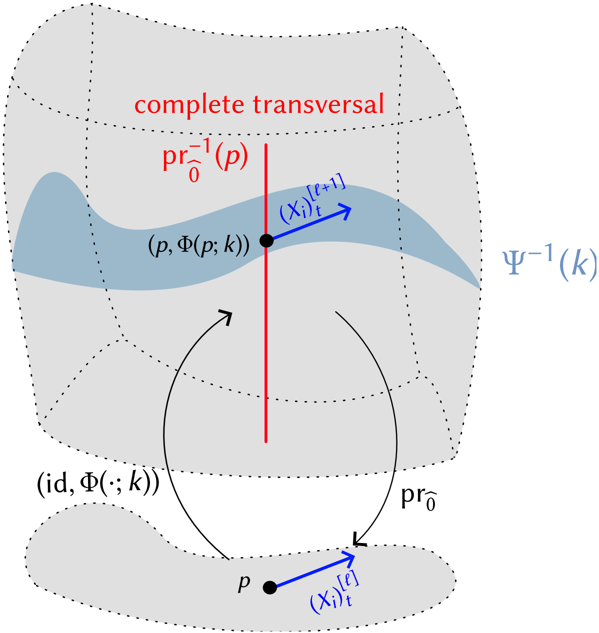
$D \subset M \times M^{\ell}$

generic points of $V^{[\ell+1]}$



$\mathrm{pr}_{\widehat{0}}(D) \subset M^{\ell}$

The global clase: Vessiot's philosophy

- $\Phi\colon M^\ell \times M \dashrightarrow M$ is **global** if every $\Phi_p\colon M \dashrightarrow M$ is a global diffeomorphism. In particular,

$$\mathrm{Dom}(\Phi) = U \times M, \qquad U \subset M^\ell.$$

- $(U, \Phi) \preceq (U', \Phi')$ if $U \subset U'$ and $\Phi = \Phi'|_{U\times M}$.

Assume that $\Phi\colon U \times M \to M$ is maximal with respect to $\preceq$

Then:

(1) $G_\Phi := \{\Phi_p\colon p \in U\}$ is a Lie group and $G_\Phi \curvearrowright M$ given by $\Phi_p \cdot k := \Phi(p; k)$ is a Lie group action whose infinitesimal generators span a VG Lie algebra for $\mathbf{X}$.

(2) The diagonal action $G_\Phi \curvearrowright M^\ell$ is such that $U \subset M^\ell$ is union of principal orbits.

Such an action is referred to as a **pretransitive Lie group action**.

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Outline

Let $V \subset \mathfrak{X}(M)$ be a VG Lie algebra.

- Consider $\mathfrak{g}$ an (abstract) Lie algebra isomorphic to V via an infinitesimal action

$$\mathscr{a} : \mathfrak{g} \to \mathfrak{X}(M), \qquad (\mathscr{a}(\mathfrak{g}) = V).$$

- Endow the trivial VB $\mathfrak{g} \times M \to M$ with
  - the VB morphism $\rho : \mathfrak{g} \times M \to TM$ defined by

$$\rho(v, u) := \mathscr{a}(v)_u,$$

  - the Lie bracket on $\Gamma(\mathfrak{g} \times M) = C^\infty(M, \mathfrak{g})$ extending the Lie bracket on constant sections.

  It follows that

$$[\alpha, f\beta] = \left(\mathcal{L}_{\rho(\alpha)} f\right) \beta + f[\alpha, \beta].$$

$$\boxed{\textbf{Action Lie algebroid } \mathfrak{g} \ltimes M \Rightarrow M}$$

This Lie algebroid is **integrable**!

- (Palais'57) $\rightsquigarrow$ Lie groupoid determined by a local Lie group action integrating V.
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The problem

Global Lie Theorem: global superposition rules $\rightsquigarrow$ VG Lie algebras formed by complete vector fields.

What about **local** superposition rules?

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Consider $U := \{u_{(i)} \neq u_{(j)}\} \subset \mathbb{R}^3$ and $p_0 := (-1, 0, 1) \in U$.

$$\mathcal{G} := \{(u_{(0)}, p) \in \mathbb{R} \times U : (u_{(0)}, p_0) \in \mathrm{Dom}(\Psi),\ (p, \Psi(u_{(0)}, p_0)) \in \mathrm{Dom}(\Phi)\}$$

Submersions $\mathrm{s}, \mathrm{t} \colon \mathcal{G} \longrightarrow \mathbb{R}$ defined by

$$\mathrm{s}(u_{(0)}, p) := u_{(0)}, \qquad \mathrm{t}(u_{(0)}, p) := \Phi(p; \Psi(u_{(0)}, p_0)).$$

Closed embedding $\mathrm{u} \colon \mathbb{R} \hookrightarrow \mathcal{G}$ given by

$$\mathrm{u}(u) := (u, p_0).$$

We have that

$$\mathrm{s} \circ \mathrm{u} = \mathrm{t} \circ \mathrm{u} = \mathrm{id}_{\mathbb{R}}.$$

Lie groupoid structure?

The target is a Möbius-type map:

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Outline

Conclusions and further work

- The 4^{th}-diagonal prolongation of $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R} \rightrightarrows \mathbb{R}$ is a Hamiltonian Lie algebroid.

C. Blohmann and A. Weinstein.
Hamiltonian Lie algebroids.
Mem. Amer. Math. Soc. **295** (2024)

Properties of the 4^{th}-diagonal prolongation of the local Lie groupoid $\mathcal{G}$?

- When is the multiplication globally associative?

R. L. Fernandes and D. Michiels.
Associativity and integrability.
Trans. Amer. Math. Soc. **373** (2020) (5057–5110).

Outline

References

J. de Lucas and C. Sardón.
A Guide to Lie Systems with Compatible Geometric Structures
(Singapore: World Scientific) 2020

J. F. Cariñena, J. Grabowski and G. Marmo.
Superposition rules, Lie theorem, and partial differential equations.
Rep. Math. Phys. **60** (2007) 237–258.

D. Blázquez-Sanz and J. J. Morales.
Local and global aspects of Lie superposition theorem.
J. Lie Theory **20** (2010) 483–517

A. Cabrera, I. Mărcuţ and M. A. Salazar.
On local integration of Lie brackets.
J. Reine Angew. Math. **2020** (2018) 267–293.

Dziękuję bardzo!

Thank you for your kind attention!

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