Lie's Theorem for superposition rules revisited

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Oscar Carballal

Based on ongoing work with J. de Lucas

Department of Algebra, Geometry and Topology
(Universidad Complutense de Madrid)

Outline

Geometric fundamentals

- **Time-dependent vector field**: $X : \mathbb{R} \times M \to TM$ such that

$$\begin{tikzcd} \mathbb{R} \times M \arrow[r, "X"] \arrow[dr] & TM \arrow[d] \\ & M \end{tikzcd} \iff \{X_t := X(t, \cdot)\}_{t \in \mathbb{R}} \subset \mathfrak{X}(M)$$

- **Associated system of ODEs**: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = X(t, x)$ in normal form.

$$\boxed{\text{Systems} \equiv \text{time-dependent vector fields}}$$

- **Superposition rule**: $\Phi : M^\ell \times M \dashrightarrow M$ satisfying
  (i) $\Phi_p := \Phi(p; \cdot) : M \dashrightarrow M$ is étale for every $p \in M^\ell$ where it is defined, and
  (ii) there exist open subsets $U \subset M^\ell$ and $W \subset M$ such that, if $x_{(1)}(t), \ldots, x_{(\ell)}(t)$ are particular solutions with $(x_{(1)}(0), \ldots, x_{(\ell)}(0)) \in U$, then every particular solution $x(t)$ with $x(0) \in W$ can be cast in the form
  $$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(\ell)}(t); k),$$
  where $k \in M$ is related to the initial conditions.

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Definition (Lie system)

A **Lie system** on M is a time-dependent vector field $X : \mathbb{R} \times M \to TM$ taking values in a *finite-dimensional Lie algebra* $V \subset \mathfrak{X}(M)$ ($\{X_t\}_{t\in\mathbb{R}} \subset V$). The Lie algebra V is a so-called **Vessiot–Guldberg Lie algebra** associated with X.

Theorem (Lie; Odzijewicz–Grundland; Cariñena–Grabowski–Marmo)

X admits a superposition rule if and only if it is a Lie system.

The Riccati equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b_0(t) + b_1(t)x + b_2(t)x^2 \rightsquigarrow X = b_0(t) \underbrace{\frac{\partial}{\partial x}}_{X_1} + b_1(t) \underbrace{x\frac{\partial}{\partial x}}_{X_2} + b_2(t) \underbrace{x^2\frac{\partial}{\partial x}}_{X_3},$$

$$V \simeq \mathfrak{sl}(2,\mathbb{R}) : \qquad [X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.$$

$$\Phi(x_{(1)}, x_{(2)}, x_{(3)}; k) := \frac{x_{(1)}(x_{(3)} - x_{(2)}) - kx_{(2)}(x_{(3)} - x_{(1)})}{(x_{(3)} - x_{(2)}) - k(x_{(3)} - x_{(1)})},$$

$$U := \left\{x_{(i)} \neq x_{(j)}\right\} \subset \mathbb{R}^3, \qquad W := \mathbb{R} - \left\{x_{(2)}(0)\right\}.$$

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Outline

Diagonal prolongations

- The ℓ^{th}**-diagonal prolongation** of a VB $\tau_E : E \longrightarrow M$ is the VB over M^ℓ with:
  - total space $E^{[\ell]} := E \times \overset{\ell}{\cdots} \times E$;
  - projection $\tau_E^{[\ell]} := \tau_E \times \overset{\ell}{\cdots} \times \tau_E$, and
  - the fibre-wise vector space structure on $E^{[\ell]}_{(x_{(1)},\dots,x_{(\ell)})} = E_{x_{(1)}} \times \overset{\ell}{\cdots} \times E_{x_{(\ell)}}$.

  It can be identified with the Whitney sum

$$E^{[\ell]} = \mathrm{pr}_1^* E \oplus \cdots \oplus \mathrm{pr}_\ell^* E \rightsquigarrow (\mathrm{T}M)^{[\ell]} = \mathrm{T}M^\ell$$

$$\Gamma(E) \ni e \mapsto e^{[\ell]} \in \Gamma\left(E^{[\ell]}\right)$$

$$e^{[\ell]}(x_{(1)}, \dots, x_{(\ell)}) := e(x_{(1)}) + \cdots + e(x_{(\ell)}).$$

- Given a VG Lie algebra $V \subset \mathfrak{X}(M)$, there exists $\ell \geq 1$ such that

$$V^{[\ell]} := \langle Y^{[\ell]} : Y \in V \rangle \simeq V$$

  possesses a modular basis; namely, a basis $\left\{X_1^{[\ell]}, \dots, X_r^{[\ell]}\right\}$ such that

$$X_1^{[\ell]} \wedge \cdots \wedge X_r^{[\ell]}$$

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Let X be a Lie system on M.

Definition (Trivialising submersion)

A **trivialising submersion** for X is a submersion $\Psi : M \times M^\ell \dashrightarrow M$ such that

(i) $\mathrm{T}\Psi\left(X_t^{[\ell+1]}|_{\mathrm{Dom}(\Psi)}\right) = 0$ for all $t \in \mathbb{R}$, and

(ii) the partial map $\Psi_p := \Psi(\cdot, p) : M \dashrightarrow M$ is étale for every $p \in M^\ell$ where it is defined.

Proposition

X admits a trivialising submersion $\Psi : M \times M^\ell \dashrightarrow M$ if and only if there exists a VG Lie algebra $V \subset \mathfrak{X}(M)$ for X such that $V^{[\ell]}$ possesses a modular basis.

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Theorem

Given $(p_0, k_0) \in M^\ell \times M$ and $x_0 \in M$, there is a 1-1 correspondence

$$\left\{ \begin{array}{c} \mathrm{germ}_{(p_0,k_0)}(\Phi) \text{ with } \Phi(p_0; k_0) = x_0 \\ \Phi : M^\ell \times M \dashrightarrow M \\ \text{superposition rule for } X \end{array} \right\} \leftrightharpoons \left\{ \begin{array}{c} \mathrm{germ}_{(x_0,p_0)}(\Psi) \text{ with } \Psi(x_0, p_0) = k_0 \\ \Psi : M \times M^\ell \dashrightarrow M \\ \text{trivialising submersion for } X \end{array} \right\}$$

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$\ell = 3$ and $\Psi : \mathbb{R} \times \mathbb{R}^3 \dashrightarrow \mathbb{R}$ given by

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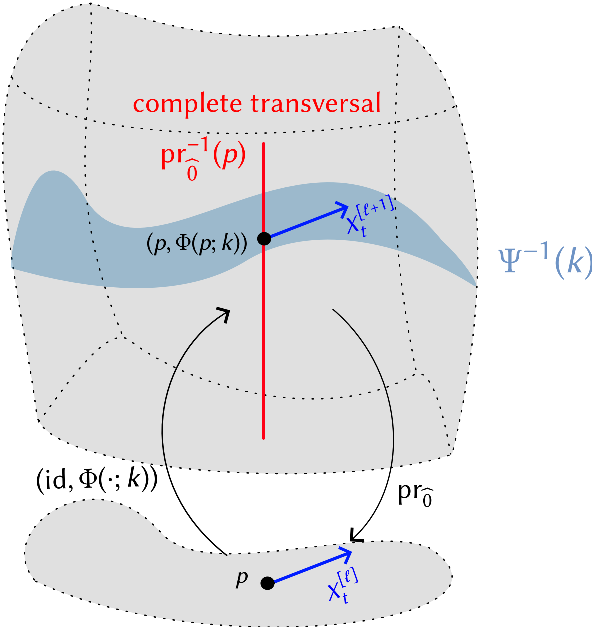
$D \subset M \times M^\ell$

generic points of $V^{[\ell+1]}$



$\mathrm{pr}_{\widehat{0}}(D) \subset M^\ell$

The global clase: Vessiot's philosophy

- $\Phi : M^\ell \times M \dashrightarrow M$ is **global** if every $\Phi_p : M \dashrightarrow M$ is a global diffeomorphism. In particular,

$$\mathrm{Dom}(\Phi) = U \times M, \qquad U \subset M^\ell.$$

- $(U, \Phi) \preceq (U', \Phi')$ if $U \subset U'$ and $\Phi = \Phi'|_{U\times M}$.

Assume that $\Phi : U \times M \longrightarrow M$ is maximal with respect to $\preceq$

Then:

(1) $G_\Phi := \{\Phi_p : p \in U\}$ is a Lie group and $G_\Phi \curvearrowright M$ given by $\Phi_p \cdot k := \Phi(p; k)$ is a Lie group action whose infinitesimal generators span a VG Lie algebra for X.

(2) The diagonal action $G_\Phi \curvearrowright M^\ell$ is such that $U \subset M^\ell$ is union of principal orbits.

Such an action is referred to as a **pretransitive Lie group action**.

Theorem (Global Lie Theorem)

X admits a global superposition rule if and only if it is induced by a pretransitive Lie group action.

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Let $V \subset \mathfrak{X}(M)$ be a VG Lie algebra.

- Consider $\mathfrak{g}$ an (abstract) Lie algebra isomorphic to V via an infinitesimal action

$$a : \mathfrak{g} \to \mathfrak{X}(M), \qquad (a(\mathfrak{g}) = V).$$

- Endow the trivial VB $\mathfrak{g} \times M \to M$ with
  - the VB morphism $\rho : \mathfrak{g} \times M \to TM$ defined by

$$\rho(v, x) := a(v)_x,$$

  - the Lie bracket on $\Gamma(\mathfrak{g} \times M) = C^\infty(M, \mathfrak{g})$ extending the Lie bracket on constant sections.

  It follows that

$$[\alpha, f\beta] = \left(\mathcal{L}_{\rho(\alpha)} f\right) \beta + f[\alpha, \beta].$$

Action Lie algebroid $\mathfrak{g} \ltimes M \rightrightarrows M$

This Lie algebroid is **integrable**!

- (Palais'57) $\rightsquigarrow$ Lie groupoid determined by a local Lie group action integrating V.
- (Dazord'97) $\rightsquigarrow$ $\mathrm{Hol}(G \times M, \mathcal{F})/G$ and $\mathrm{Mon}(G \times M, \mathcal{F})/G$, where G is the 1-connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$, integrate $\mathfrak{g} \times M \rightrightarrows M$.

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$$[\alpha, f\beta] = \left(\mathcal{L}_{\rho(\alpha)} f\right) \beta + f[\alpha, \beta].$$

Action Lie algebroid $\mathfrak{g} \ltimes M \Rightarrow M$

This Lie algebroid is **integrable**!

- (Palais'57) $\rightsquigarrow$ Lie groupoid determined by a local Lie group action integrating V.
- (Dazord'97) $\rightsquigarrow$ $\mathrm{Hol}(G \times M, \mathcal{F})/G$ and $\mathrm{Mon}(G \times M, \mathcal{F})/G$, where G is the 1-connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$, integrate $\mathfrak{g} \times M \Rightarrow M$.

Let $V \subset \mathfrak{X}(M)$ be a VG Lie algebra.

- Consider $\mathfrak{g}$ an (abstract) Lie algebra isomorphic to V via an infinitesimal action

$$a : \mathfrak{g} \to \mathfrak{X}(M), \qquad (a(\mathfrak{g}) = V).$$

- Endow the trivial VB $\mathfrak{g} \times M \to M$ with
  - the VB morphism $\rho : \mathfrak{g} \times M \to TM$ defined by

$$\rho(v, x) := a(v)_x,$$

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Global Lie Theorem: global superposition rules $\rightsquigarrow$ VG Lie algebras formed by complete vector fields.

What about **local** superposition rules?

The Riccati equation

Superposition rule $\Phi : \mathbb{R}^3 \times \mathbb{R} \dashrightarrow \mathbb{R}$ and trivialising submersion $\Psi : \mathbb{R} \times \mathbb{R}^3 \dashrightarrow \mathbb{R}$ given by

$$\Phi(p; k) := \frac{x_{(1)}(x_{(3)} - x_{(2)}) - kx_{(2)}(x_{(3)} - x_{(1)})}{(x_{(3)} - x_{(2)}) - k(x_{(3)} - x_{(1)})},$$

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They satisfy

$$\Psi(\Phi(p; k), p) = k, \qquad \Phi(p; \Psi(x_{(0)}, p)) = x_{(0)}.$$

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Conclusions and further work

- The 4^{th}-diagonal prolongation of $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R} \rightrightarrows \mathbb{R}$ is a Hamiltonian Lie algebroid.

C. Blohmann and A. Weinstein.
Hamiltonian Lie algebroids.
Mem. Amer. Math. Soc. **295** (2024)

  Properties of the 4^{th}-diagonal prolongation of the local Lie groupoid $\mathcal{G}$?

- Multiplicative forms on $\mathcal{G}$ and their corresponding infinitesimally multiplicative structures on the action Lie algebroid.

Outline

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Thank you for your kind attention!

✉ oscarbal@ucm.es