

Contact Lie systems on three-dimensional Riemannian and Lorentzian spaces of constant curvature

RSME's VIIth Congress of Young Researchers

OSCAR CARBALLAL

Joint work with R. Campoamor-Stursberg (UCM) and F. J. Herranz (UBU)

Department of Algebra, Geometry and Topology (UCM)



① Lie systems

Compatible geometric structures

② 3D CK spaces

③ Contact Lie systems on 3D CK spaces

④ Conclusions and open problems

⑤ References

Outline

- 1 Lie systems
Compatible geometric structures
- 2 3D CK spaces
- 3 Contact Lie systems on 3D CK spaces
- 4 Conclusions and open problems
- 5 References

Consider the first-order *Riccati equation* on $\mathbb{R}$:

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2, \quad a_\alpha \in C^\infty(\mathbb{R}).$$

- There is no general method for obtaining its general solution!
- Lie, Vessiot and Guldberg (among others) noticed that the general solution $x(t)$ can be expressed as

$$x(t) = \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) + kx_{(2)}(t)(x_{(1)}(t) - x_{(3)}(t))}{x_{(3)}(t) - x_{(2)}(t) + k(x_{(1)}(t) - x_{(3)}(t))}$$

with $x_{(i)}(t)$ a particular solution and $k \in \mathbb{R} \cup \{\infty\}$.

Such expression is known as a *superposition rule*.

Consider the first-order *Riccati equation* on $\mathbb{R}$:

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2, \quad a_\alpha \in C^\infty(\mathbb{R}).$$

- There is no general method for obtaining its general solution!
- Lie, Vessiot and Guldberg (among others) noticed that the general solution $x(t)$ can be expressed as

$$x(t) = \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) + kx_{(2)}(t)(x_{(1)}(t) - x_{(3)}(t))}{x_{(3)}(t) - x_{(2)}(t) + k(x_{(1)}(t) - x_{(3)}(t))}$$

with $x_{(i)}(t)$ a particular solution and $k \in \mathbb{R} \cup \{\infty\}$.

Such expression is known as a *superposition rule*.

Consider the first-order *Riccati equation* on $\mathbb{R}$:

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2, \quad a_\alpha \in C^\infty(\mathbb{R}).$$

- There is no general method for obtaining its general solution!
- Lie, Vessiot and Guldberg (among others) noticed that the general solution $x(t)$ can be expressed as

$$x(t) = \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) + kx_{(2)}(t)(x_{(1)}(t) - x_{(3)}(t))}{x_{(3)}(t) - x_{(2)}(t) + k(x_{(1)}(t) - x_{(3)}(t))}$$

with $x_{(i)}(t)$ a particular solution and $k \in \mathbb{R} \cup \{\infty\}$.

Such expression is known as a *superposition rule*.

Consider a first-order system of ODEs on a manifold M in normal form:

$$\frac{dx^i}{dt} = \psi^i(t, x), \quad x \in M, \quad 1 \leq i \leq n := \dim M.$$

Its solutions are the integral curves of the t -dependent vector field

$$\mathbf{X} : \mathbb{R} \times M \rightarrow TM$$

locally given by $\mathbf{X}(t, x) = \psi^i(t, x) \frac{\partial}{\partial x^i}$.

$$\mathbf{X} \rightsquigarrow \{\mathbf{X}_t\}_{t \in \mathbb{R}} \subset \mathfrak{X}(M) \text{ such that } \mathbf{X}_t(x) := \mathbf{X}(t, x).$$

$$\left\{ \begin{array}{c} \text{1st order ODEs systems} \\ \text{in normal form} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} t\text{-dependent} \\ \text{vector fields} \end{array} \right\}$$

Consider a first-order system of ODEs on a manifold M in normal form:

$$\frac{dx^i}{dt} = \psi^i(t, x), \quad x \in M, \quad 1 \leq i \leq n := \dim M.$$

Its solutions are the integral curves of the t -dependent vector field

$$\mathbf{X} : \mathbb{R} \times M \rightarrow TM$$

locally given by $\mathbf{X}(t, x) = \psi^i(t, x) \frac{\partial}{\partial x^i}$.

$$\mathbf{X} \rightsquigarrow \{\mathbf{X}_t\}_{t \in \mathbb{R}} \subset \mathfrak{X}(M) \text{ such that } \mathbf{X}_t(x) := \mathbf{X}(t, x).$$

$$\left\{ \begin{array}{c} \text{1st order ODEs systems} \\ \text{in normal form} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} t\text{-dependent} \\ \text{vector fields} \end{array} \right\}$$

Consider a first-order system of ODEs on a manifold M in normal form:

$$\frac{dx^i}{dt} = \psi^i(t, x), \quad x \in M, \quad 1 \leq i \leq n := \dim M.$$

Its solutions are the integral curves of the t -dependent vector field

$$\mathbf{X} : \mathbb{R} \times M \rightarrow TM$$

locally given by $\mathbf{X}(t, x) = \psi^i(t, x) \frac{\partial}{\partial x^i}$.

$$\mathbf{X} \rightsquigarrow \{\mathbf{X}_t\}_{t \in \mathbb{R}} \subset \mathfrak{X}(M) \text{ such that } \mathbf{X}_t(x) := \mathbf{X}(t, x).$$

$$\left\{ \begin{array}{c} \text{1st order ODEs systems} \\ \text{in normal form} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} t\text{-dependent} \\ \text{vector fields} \end{array} \right\}$$

Consider a first-order system of ODEs on a manifold M in normal form:

$$\frac{dx^i}{dt} = \psi^i(t, x), \quad x \in M, \quad 1 \leq i \leq n := \dim M.$$

Its solutions are the integral curves of the t -dependent vector field

$$\mathbf{X} : \mathbb{R} \times M \rightarrow TM$$

locally given by $\mathbf{X}(t, x) = \psi^i(t, x) \frac{\partial}{\partial x^i}$.

$$\mathbf{X} \rightsquigarrow \{\mathbf{X}_t\}_{t \in \mathbb{R}} \subset \mathfrak{X}(M) \text{ such that } \mathbf{X}_t(x) := \mathbf{X}(t, x).$$

$$\left\{ \begin{array}{c} \text{1st order ODEs systems} \\ \text{in normal form} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} t\text{-dependent} \\ \text{vector fields} \end{array} \right\}$$

A *superposition rule* for $\mathbf{X}$ is a t -independent map

$$\Psi : M^s \times M \rightarrow M$$

such that the general solution $\mathbf{x}(t)$ of $\mathbf{X}$ can be expressed as

$$\mathbf{x}(t) = \Psi(\mathbf{x}_{(1)}(t), \dots, \mathbf{x}_{(s)}(t); k),$$

where $\mathbf{x}_{(1)}(t), \dots, \mathbf{x}_{(s)}(t)$ is a generic family of particular solutions and $k \in M$ is a point related to the initial conditions.

Definition (Lie system)

$\mathbf{X}$ is a *Lie system* if it admits a (local) superposition rule.

A *superposition rule* for $\mathbf{X}$ is a t -independent map

$$\Psi : M^s \times M \rightarrow M$$

such that the general solution $\mathbf{x}(t)$ of $\mathbf{X}$ can be expressed as

$$\mathbf{x}(t) = \Psi(\mathbf{x}_{(1)}(t), \dots, \mathbf{x}_{(s)}(t); k),$$

where $\mathbf{x}_{(1)}(t), \dots, \mathbf{x}_{(s)}(t)$ is a generic family of particular solutions and $k \in M$ is a point related to the initial conditions.

Definition (Lie system)

$\mathbf{X}$ is a *Lie system* if it admits a (local) superposition rule.

Theorem (Lie–Scheffers)

$\mathbf{X}$ is a Lie system if and only if it admits the decomposition

$$\mathbf{X}(t, \mathbf{x}) = \sum_{\alpha=1}^r b_{\alpha}(t) \mathbf{X}_{\alpha}$$

for some functions $b_1, \dots, b_r \in C^{\infty}(\mathbb{R})$ and vector fields $\mathbf{X}_1, \dots, \mathbf{X}_r$ on M spanning an r -dimensional real Lie algebra $V^{\mathbf{X}}$.

$V^{\mathbf{X}}$ is usually called a *Vessiot–Guldberg (VG) Lie algebra*.

Theorem (Lie–Scheffers)

$\mathbf{X}$ is a Lie system if and only if it admits the decomposition

$$\mathbf{X}(t, \mathbf{x}) = \sum_{\alpha=1}^r b_{\alpha}(t) \mathbf{X}_{\alpha}$$

for some functions $b_1, \dots, b_r \in C^{\infty}(\mathbb{R})$ and vector fields $\mathbf{X}_1, \dots, \mathbf{X}_r$ on M spanning an r -dimensional real Lie algebra V^X .

V^X is usually called a *Vessiot–Guldberg (VG) Lie algebra*.

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 \rightsquigarrow \mathbf{X} = a_0(t)\mathbf{X}_1 + a_1(t)\mathbf{X}_2 + a_2(t)\mathbf{X}_3$$

where the vector fields

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = x \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = x^2 \frac{\partial}{\partial x}$$

have commutation rules

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3,$$

and therefore span a VG Lie algebra $V^X \simeq \mathfrak{sl}(2, \mathbb{R})$.

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 \rightsquigarrow \mathbf{X} = a_0(t)\mathbf{X}_1 + a_1(t)\mathbf{X}_2 + a_2(t)\mathbf{X}_3$$

where the vector fields

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = x \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = x^2 \frac{\partial}{\partial x}$$

have commutation rules

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3,$$

and therefore span a VG Lie algebra $V^X \simeq \mathfrak{sl}(2, \mathbb{R})$.

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 \rightsquigarrow \mathbf{X} = a_0(t)\mathbf{X}_1 + a_1(t)\mathbf{X}_2 + a_2(t)\mathbf{X}_3$$

where the vector fields

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = x \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = x^2 \frac{\partial}{\partial x}$$

have commutation rules

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3,$$

and therefore span a VG Lie algebra $V^X \simeq \mathfrak{sl}(2, \mathbb{R})$.

(M, ω)	(M, η)
$\dim M = 2n, d\omega = 0, \omega^n \neq 0$	$\dim M = 2n + 1, \eta \wedge (d\eta)^n \neq 0$
$\iota_{\mathbf{X}_f} \omega = df$	$\iota_{\mathcal{R}} \eta = 1, \iota_{\mathcal{R}} d\eta = 0$
$\{f, g\}_{\omega} = \omega(\mathbf{X}_f, \mathbf{X}_g)$	$\iota_{\mathbf{X}_f} \eta = -f, \iota_{\mathbf{X}_f} d\eta = df - (\mathcal{R}f)\eta$
	$\{f, g\}_{\eta} = \mathbf{X}_f g + g\mathcal{R}f$

Definition

- A Lie system $\mathbf{X}$ on (M, ω) is a *Lie-Hamilton system* if $V^{\mathbf{X}}$ is formed by Hamiltonian vector fields relative to ω .
- A Lie system $\mathbf{X}$ on (M, η) is a *contact Lie system* if $V^{\mathbf{X}}$ is formed by contact Hamiltonian vector fields relative to η . If $\mathcal{R}$ is a Lie symmetry of $V^{\mathbf{X}}$, then $\mathbf{X}$ is called a contact Lie system *of Liouville type*.

(M, ω)	(M, η)
$\dim M = 2n, d\omega = 0, \omega^n \neq 0$	$\dim M = 2n + 1, \eta \wedge (d\eta)^n \neq 0$
$\iota_{\mathbf{X}_f} \omega = df$	$\iota_{\mathcal{R}} \eta = 1, \iota_{\mathcal{R}} d\eta = 0$
$\{f, g\}_{\omega} = \omega(\mathbf{X}_f, \mathbf{X}_g)$	$\iota_{\mathbf{X}_f} \eta = -f, \iota_{\mathbf{X}_f} d\eta = df - (\mathcal{R}f)\eta$
	$\{f, g\}_{\eta} = \mathbf{X}_f g + g\mathcal{R}f$

Definition

- A Lie system $\mathbf{X}$ on (M, ω) is a *Lie–Hamilton system* if $V^{\mathbf{X}}$ is formed by Hamiltonian vector fields relative to ω .
- A Lie system $\mathbf{X}$ on (M, η) is a *contact Lie system* if $V^{\mathbf{X}}$ is formed by contact Hamiltonian vector fields relative to η . If $\mathcal{R}$ is a Lie symmetry of $V^{\mathbf{X}}$, then $\mathbf{X}$ is called a contact Lie system *of Liouville type*.

(M, ω)	(M, η)
$\dim M = 2n, d\omega = 0, \omega^n \neq 0$	$\dim M = 2n + 1, \eta \wedge (d\eta)^n \neq 0$
$\iota_{\mathbf{X}_f} \omega = df$	$\iota_{\mathcal{R}} \eta = 1, \iota_{\mathcal{R}} d\eta = 0$
$\{f, g\}_{\omega} = \omega(\mathbf{X}_f, \mathbf{X}_g)$	$\iota_{\mathbf{X}_f} \eta = -f, \iota_{\mathbf{X}_f} d\eta = df - (\mathcal{R}f)\eta$
	$\{f, g\}_{\eta} = \mathbf{X}_f g + g\mathcal{R}f$

Definition

- A Lie system $\mathbf{X}$ on (M, ω) is a *Lie–Hamilton system* if $V^{\mathbf{X}}$ is formed by Hamiltonian vector fields relative to ω .
- A Lie system $\mathbf{X}$ on (M, η) is a *contact Lie system* if $V^{\mathbf{X}}$ is formed by contact Hamiltonian vector fields relative to η . If $\mathcal{R}$ is a Lie symmetry of $V^{\mathbf{X}}$, then $\mathbf{X}$ is called a contact Lie system *of Liouville type*.

The integral curves of $\mathbf{X}$ are the solutions of the Hamilton equations of a *t-dependent Hamiltonian*

$$h : \mathbb{R} \times M \rightarrow \mathbb{R}, \quad h(t, x) := h_t(x),$$

with $\mathbf{X}_t$ the Hamiltonian vector field of h_t .

Definition (Lie–Hamilton algebra of $\mathbf{X}$)

- $\mathcal{H}_\omega := \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\omega)$.
- $\mathcal{H}_\eta := \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\eta)$.



J. de Lucas and C. Sardón.

A Guide to Lie Systems with Compatible Geometric Structures.
(Singapore: World Scientific) 2020.

The integral curves of $\mathbf{X}$ are the solutions of the Hamilton equations of a *t-dependent Hamiltonian*

$$h : \mathbb{R} \times M \rightarrow \mathbb{R}, \quad h(t, x) := h_t(x),$$

with $\mathbf{X}_t$ the Hamiltonian vector field of h_t .

Definition (Lie–Hamilton algebra of $\mathbf{X}$)

- $\mathcal{H}_\omega := \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\omega)$.
- $\mathcal{H}_\eta := \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\eta)$.



J. de Lucas and C. Sardón.

A Guide to Lie Systems with Compatible Geometric Structures.
(Singapore: World Scientific) 2020.

The integral curves of $\mathbf{X}$ are the solutions of the Hamilton equations of a *t-dependent Hamiltonian*

$$h : \mathbb{R} \times M \rightarrow \mathbb{R}, \quad h(t, x) := h_t(x),$$

with $\mathbf{X}_t$ the Hamiltonian vector field of h_t .

Definition (Lie–Hamilton algebra of $\mathbf{X}$)

- $\mathcal{H}_\omega := \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\omega)$.
- $\mathcal{H}_\eta := \text{Lie}(\{h_t\}_{t \in \mathbb{R}}, \{\cdot, \cdot\}_\eta)$.



J. de Lucas and C. Sardón.

A Guide to Lie Systems with Compatible Geometric Structures.
(Singapore: World Scientific) 2020.

Outline

- 1 Lie systems
Compatible geometric structures
- 2 3D CK spaces
- 3 Contact Lie systems on 3D CK spaces
- 4 Conclusions and open problems
- 5 References

- $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ the Lie group of isometries of

$\mathbb{I}_{\kappa} := \mathrm{diag}(1, \kappa_1, \kappa_1 \kappa_2, \kappa_1 \kappa_2 \kappa_3)$ on $\mathbb{R}^4$, with $\kappa_i \in \{0, \pm 1\}$.

- H_0 the isotropy subgroup of the point $O := (1, 0, 0, 0)$.

Definition (3D Cayley–Klein space)

$$\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3 := \mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4) / H_0.$$

- g_{κ} the projection of metric on $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ obtained by right translation of $\mathbb{I}_{\kappa}$.

κ_1 is a (sectional) *curvature* and κ_2, κ_3 determine the *signature* through $\mathrm{diag}(1, \kappa_2, \kappa_2 \kappa_3)$.

- $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ the Lie group of isometries of

$\mathbb{I}_{\kappa} := \mathrm{diag}(1, \kappa_1, \kappa_1 \kappa_2, \kappa_1 \kappa_2 \kappa_3)$ on $\mathbb{R}^4$, with $\kappa_i \in \{0, \pm 1\}$.

- H_0 the isotropy subgroup of the point $O := (1, 0, 0, 0)$.

Definition (3D Cayley–Klein space)

$$\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3 := \mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4) / H_0.$$

- g_{κ} the projection of metric on $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ obtained by right translation of $\mathbb{I}_{\kappa}$.

κ_1 is a (sectional) *curvature* and κ_2, κ_3 determine the *signature* through $\mathrm{diag}(1, \kappa_2, \kappa_2 \kappa_3)$.

- $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ the Lie group of isometries of

$\mathbb{I}_{\kappa} := \mathrm{diag}(1, \kappa_1, \kappa_1 \kappa_2, \kappa_1 \kappa_2 \kappa_3)$ on $\mathbb{R}^4$, with $\kappa_i \in \{0, \pm 1\}$.

- H_0 the isotropy subgroup of the point $O := (1, 0, 0, 0)$.

Definition (3D Cayley–Klein space)

$$\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3 := \mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)/H_0.$$

- g_{κ} the projection of metric on $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ obtained by right translation of $\mathbb{I}_{\kappa}$.

κ_1 is a (sectional) *curvature* and κ_2, κ_3 determine the *signature* through $\mathrm{diag}(1, \kappa_2, \kappa_2 \kappa_3)$.

- $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ the Lie group of isometries of

$\mathbb{I}_{\kappa} := \mathrm{diag}(1, \kappa_1, \kappa_1 \kappa_2, \kappa_1 \kappa_2 \kappa_3)$ on $\mathbb{R}^4$, with $\kappa_i \in \{0, \pm 1\}$.

- H_0 the isotropy subgroup of the point $O := (1, 0, 0, 0)$.

Definition (3D Cayley–Klein space)

$$\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3 := \mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)/H_0.$$

- g_{κ} the projection of metric on $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ obtained by right translation of $\mathbb{I}_{\kappa}$.

κ_1 is a (sectional) *curvature* and κ_2, κ_3 determine the *signature* through $\mathrm{diag}(1, \kappa_2, \kappa_2 \kappa_3)$.

- $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ the Lie group of isometries of

$\mathbb{I}_{\kappa} := \mathrm{diag}(1, \kappa_1, \kappa_1 \kappa_2, \kappa_1 \kappa_2 \kappa_3)$ on $\mathbb{R}^4$, with $\kappa_i \in \{0, \pm 1\}$.

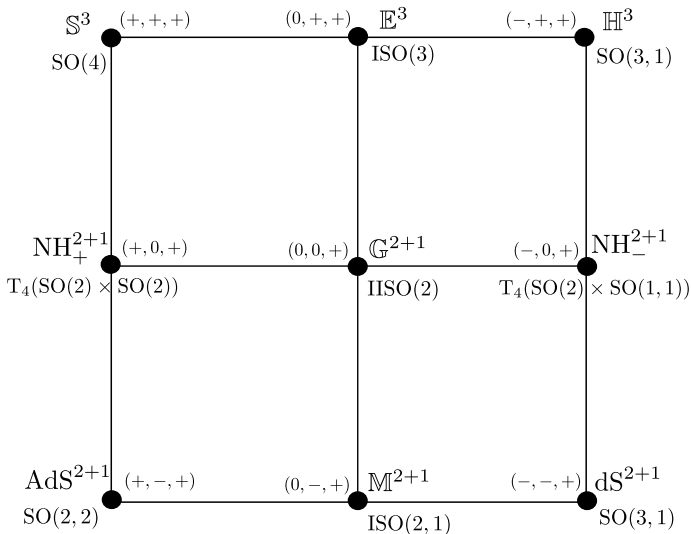
- H_0 the isotropy subgroup of the point $O := (1, 0, 0, 0)$.

Definition (3D Cayley–Klein space)

$$\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3 := \mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)/H_0.$$

- g_{κ} the projection of metric on $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ obtained by right translation of $\mathbb{I}_{\kappa}$.

κ_1 is a (sectional) *curvature* and κ_2, κ_3 determine the *signature* through $\mathrm{diag}(1, \kappa_2, \kappa_2 \kappa_3)$.

KINEMATICAL SPACES $\mathbb{S}^3_{[\kappa_1], \kappa_2, +}$ *à la Bacry & Lévy-Leblond*

- $\mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3}$ is diffeomorphic to the connected component of

$$\{(x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2 + \kappa_1\kappa_2\kappa_3(x^3)^2 = 1\} = \mathbb{I}_{\kappa}^{-1}(1)$$

containing $O = (1, 0, 0, 0)$.

- Exact symplectic manifold $(\mathbb{R}_0^4 := \mathbb{R}^4 - \{0\}, \omega = dx^0 \wedge dx^1 + dx^2 \wedge dx^3)$ and Liouville vector field

$$\Delta := \frac{1}{2} \left(x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} \right) \rightsquigarrow \omega = d\iota_{\Delta}\omega.$$

- Δ is transversal to $i : \mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3} \hookrightarrow \mathbb{R}_0^4$, so

$$\eta_{\kappa} := i^* \iota_{\Delta} \omega$$

is a contact form on $\mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3}$.

- $\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3$ is diffeomorphic to the connected component of

$$\{(x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2 + \kappa_1\kappa_2\kappa_3(x^3)^2 = 1\} = \mathbb{I}_{\kappa}^{-1}(1)$$

containing $O = (1, 0, 0, 0)$.

- Exact symplectic manifold $(\mathbb{R}_0^4 := \mathbb{R}^4 - \{0\}, \omega = dx^0 \wedge dx^1 + dx^2 \wedge dx^3)$ and Liouville vector field

$$\Delta := \frac{1}{2} \left(x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} \right) \rightsquigarrow \omega = d\iota_{\Delta} \omega.$$

- Δ is transversal to $i : \mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3 \hookrightarrow \mathbb{R}_0^4$, so

$$\eta_{\kappa} := i^* \iota_{\Delta} \omega$$

is a contact form on $\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3$.

- $\mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3}$ is diffeomorphic to the connected component of

$$\{(x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2 + \kappa_1\kappa_2\kappa_3(x^3)^2 = 1\} = \mathbb{I}_{\kappa}^{-1}(1)$$

containing $O = (1, 0, 0, 0)$.

- Exact symplectic manifold $(\mathbb{R}_0^4 := \mathbb{R}^4 - \{0\}, \omega = dx^0 \wedge dx^1 + dx^2 \wedge dx^3)$ and Liouville vector field

$$\Delta := \frac{1}{2} \left(x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} \right) \rightsquigarrow \omega = d\iota_{\Delta} \omega.$$

- Δ is transversal to $i : \mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3} \hookrightarrow \mathbb{R}_0^4$, so

$$\eta_{\kappa} := i^* \iota_{\Delta} \omega$$

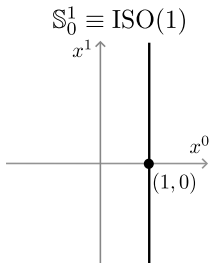
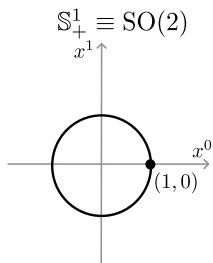
is a contact form on $\mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3}$.

Theorem

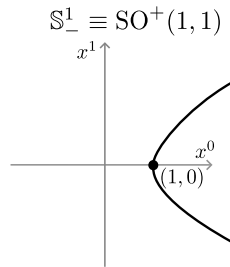
There exists an invariant $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ -connection ∇ on $\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3$ such that $\nabla_{\mathcal{R}_\kappa} \mathcal{R}_\kappa = 0$.

The integral curves of $\mathcal{R}_\kappa$ define a codimension one foliation $\mathcal{F}_\kappa$ of $\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3$ by geodesics.

$$\mathbb{S}_{[\kappa_1]}^1 := \{(x^0) + \kappa_1(x^1)^2 = 1, x^0 > 0\}$$



$\xrightarrow[\text{isomorphic}]{\sim}$

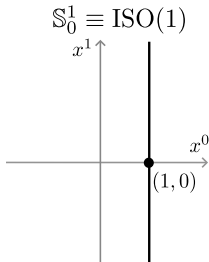
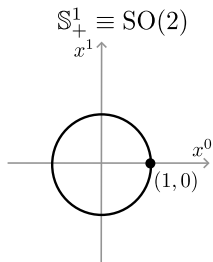


Theorem

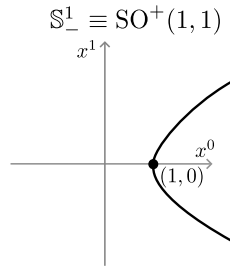
There exists an invariant $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ -connection ∇ on $\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3$ such that $\nabla_{\mathcal{R}_\kappa} \mathcal{R}_\kappa = 0$.

The integral curves of $\mathcal{R}_\kappa$ define a codimension one foliation $\mathcal{F}_\kappa$ of $\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3$ by geodesics.

$$\mathbb{S}_{[\kappa_1]}^1 := \{(x^0) + \kappa_1(x^1)^2 = 1, x^0 > 0\}$$



$\xrightarrow[\text{isomorphic}]{\sim}$

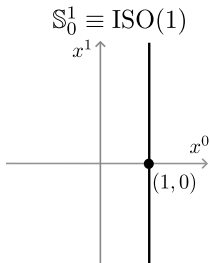
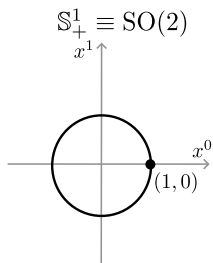


Theorem

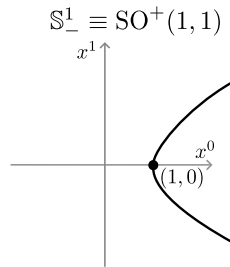
There exists an invariant $\mathrm{SO}_{\kappa_1, \kappa_2, \kappa_3}(4)$ -connection ∇ on $\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3$ such that $\nabla_{\mathcal{R}_\kappa} \mathcal{R}_\kappa = 0$.

The integral curves of $\mathcal{R}_\kappa$ define a codimension one foliation $\mathcal{F}_\kappa$ of $\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3$ by geodesics.

$$\mathbb{S}_{[\kappa_1]}^1 := \{(x^0) + \kappa_1(x^1)^2 = 1, x^0 > 0\}$$



$\xrightarrow[\text{isomorphic}]{\sim}$



Theorem

The foliation $\mathcal{F}_\kappa$ on $\mathbb{S}^3_{[\kappa_1], \kappa_2, +}$ is regular when $(\kappa_1, \kappa_2) \neq (-1, -1)$, inducing an $\mathbb{S}^1_{[\kappa_1]}$ -principal bundle $\mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3} \rightarrow \mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3} / \mathcal{R}_\kappa$.

From this one recovers the Hopf fibrations

$$\mathbb{S}^3 \rightarrow \mathbb{S}^2, \quad \text{AdS}^{2+1} \rightarrow \mathbb{CH}^1.$$

Theorem

The foliation $\mathcal{F}_\kappa$ on $\mathbb{S}^3_{[\kappa_1], \kappa_2, +}$ is regular when $(\kappa_1, \kappa_2) \neq (-1, -1)$, inducing an $\mathbb{S}^1_{[\kappa_1]}$ -principal bundle $\mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3} \rightarrow \mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3} / \mathcal{R}_\kappa$.

From this one recovers the Hopf fibrations

$$\mathbb{S}^3 \rightarrow \mathbb{S}^2, \quad \text{AdS}^{2+1} \rightarrow \mathbb{CH}^1.$$

Outline

- 1 Lie systems
Compatible geometric structures
- 2 3D CK spaces
- 3 Contact Lie systems on 3D CK spaces
- 4 Conclusions and open problems
- 5 References

- The Hamiltonian functions $h_\alpha \in C^\infty(\mathbb{R}_0^4)$ given by

$$\begin{aligned} h_1 &= x^0 x^1, & h_2 &= x^0 x^3, & h_3 &= x^1 x^2, & h_4 &= x^2 x^3, & h_5 &= \frac{1}{2}(x^0)^2, \\ h_6 &= x^0 x^2, & h_7 &= \frac{1}{2}(x^2)^2, & h_8 &= \frac{1}{2}(x^1)^2, & h_9 &= x^1 x^3, & h_{10} &= \frac{1}{2}(x^3)^2, \end{aligned}$$

span a Lie–Hamilton algebra $\mathcal{H}_\omega \simeq \mathfrak{sp}(4, \mathbb{R})$ with respect to $\{\cdot, \cdot\}_\omega$.

- Lie–Hamilton system on $(\mathbb{R}_0^4, \omega)$ with t -dependent Hamiltonian $h := \sum_{\alpha=1}^{10} b_\alpha(t) h_\alpha$.
- The contact Hamiltonian functions $h_{\kappa, \alpha} := i^* h_\alpha \in C^\infty(\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3)$ span a Lie–Hamilton algebra $\mathcal{H}_{\eta_\kappa} \simeq \mathfrak{sp}(4, \mathbb{R})$ with respect to $\{\cdot, \cdot\}_{\eta_\kappa}$.
- Contact Lie system on $(\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3, \eta_\kappa)$ associated to $h_\kappa = \sum_{\alpha=1}^{10} b_\alpha(t) h_{\kappa, \alpha}$.

- The Hamiltonian functions $h_\alpha \in C^\infty(\mathbb{R}_0^4)$ given by

$$\begin{aligned} h_1 &= x^0 x^1, & h_2 &= x^0 x^3, & h_3 &= x^1 x^2, & h_4 &= x^2 x^3, & h_5 &= \frac{1}{2}(x^0)^2, \\ h_6 &= x^0 x^2, & h_7 &= \frac{1}{2}(x^2)^2, & h_8 &= \frac{1}{2}(x^1)^2, & h_9 &= x^1 x^3, & h_{10} &= \frac{1}{2}(x^3)^2, \end{aligned}$$

span a Lie–Hamilton algebra $\mathcal{H}_\omega \simeq \mathfrak{sp}(4, \mathbb{R})$ with respect to $\{\cdot, \cdot\}_\omega$.

- Lie–Hamilton system on $(\mathbb{R}_0^4, \omega)$ with t -dependent Hamiltonian $h := \sum_{\alpha=1}^{10} b_\alpha(t) h_\alpha$.
- The contact Hamiltonian functions $h_{\kappa, \alpha} := i^* h_\alpha \in C^\infty(\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3)$ span a Lie–Hamilton algebra $\mathcal{H}_{\eta_\kappa} \simeq \mathfrak{sp}(4, \mathbb{R})$ with respect to $\{\cdot, \cdot\}_{\eta_\kappa}$.
- Contact Lie system on $(\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3, \eta_\kappa)$ associated to $h_\kappa = \sum_{\alpha=1}^{10} b_\alpha(t) h_{\kappa, \alpha}$.

- The Hamiltonian functions $h_\alpha \in C^\infty(\mathbb{R}_0^4)$ given by

$$\begin{aligned} h_1 &= x^0 x^1, & h_2 &= x^0 x^3, & h_3 &= x^1 x^2, & h_4 &= x^2 x^3, & h_5 &= \frac{1}{2}(x^0)^2, \\ h_6 &= x^0 x^2, & h_7 &= \frac{1}{2}(x^2)^2, & h_8 &= \frac{1}{2}(x^1)^2, & h_9 &= x^1 x^3, & h_{10} &= \frac{1}{2}(x^3)^2, \end{aligned}$$

span a Lie–Hamilton algebra $\mathcal{H}_\omega \simeq \mathfrak{sp}(4, \mathbb{R})$ with respect to $\{\cdot, \cdot\}_\omega$.

- Lie–Hamilton system on $(\mathbb{R}_0^4, \omega)$ with t -dependent Hamiltonian $h := \sum_{\alpha=1}^{10} b_\alpha(t) h_\alpha$.
- The contact Hamiltonian functions $h_{\kappa, \alpha} := i^* h_\alpha \in C^\infty(\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3)$ span a Lie–Hamilton algebra $\mathcal{H}_{\eta_\kappa} \simeq \mathfrak{sp}(4, \mathbb{R})$ with respect to $\{\cdot, \cdot\}_{\eta_\kappa}$.
- Contact Lie system on $(\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3, \eta_\kappa)$ associated to $h_\kappa = \sum_{\alpha=1}^{10} b_\alpha(t) h_{\kappa, \alpha}$.

- The Hamiltonian functions $h_\alpha \in C^\infty(\mathbb{R}_0^4)$ given by

$$\begin{aligned} h_1 &= x^0 x^1, & h_2 &= x^0 x^3, & h_3 &= x^1 x^2, & h_4 &= x^2 x^3, & h_5 &= \frac{1}{2}(x^0)^2, \\ h_6 &= x^0 x^2, & h_7 &= \frac{1}{2}(x^2)^2, & h_8 &= \frac{1}{2}(x^1)^2, & h_9 &= x^1 x^3, & h_{10} &= \frac{1}{2}(x^3)^2, \end{aligned}$$

span a Lie–Hamilton algebra $\mathcal{H}_\omega \simeq \mathfrak{sp}(4, \mathbb{R})$ with respect to $\{\cdot, \cdot\}_\omega$.

- Lie–Hamilton system on $(\mathbb{R}_0^4, \omega)$ with t -dependent Hamiltonian $h := \sum_{\alpha=1}^{10} b_\alpha(t) h_\alpha$.
- The contact Hamiltonian functions $h_{\kappa, \alpha} := i^* h_\alpha \in C^\infty(\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3)$ span a Lie–Hamilton algebra $\mathcal{H}_{\eta_\kappa} \simeq \mathfrak{sp}(4, \mathbb{R})$ with respect to $\{\cdot, \cdot\}_{\eta_\kappa}$.
- Contact Lie system on $(\mathbb{S}_{[\kappa_1], \kappa_2, \kappa_3}^3, \eta_\kappa)$ associated to $h_\kappa = \sum_{\alpha=1}^{10} b_\alpha(t) h_{\kappa, \alpha}$.

- $\mathbb{S}^3 = \mathbb{S}^3_{[+],+,+}$ and $\text{AdS}^{2+1} = \mathbb{S}^3_{[+],-,+}$ are *Sasakian manifolds*. This is,

$$\omega = d(s^2 \eta_{\kappa}), \quad ds \otimes ds + s^2 g_{\kappa}$$

define Kähler structures on the cones $\mathbb{S}^3 \times \mathbb{R}_+$ and $\text{AdS}^{2+1} \times \mathbb{R}_+$.

Proposition

Let (M, η, g) be a Sasakian manifold and $\mathbf{K}$ a Killing vector field. Then, $\iota_{\mathbf{K}}\eta$ is a first integral of $\mathcal{R}$.

From this one can obtain contact Lie systems of Liouville type on $\mathbb{S}^3$ and AdS^{2+1} .

- $\mathbb{S}^3 = \mathbb{S}^3_{[+],+,+}$ and $\text{AdS}^{2+1} = \mathbb{S}^3_{[+],-,+}$ are *Sasakian manifolds*. This is,

$$\omega = d(s^2\eta_{\kappa}), \quad ds \otimes ds + s^2 g_{\kappa}$$

define Kähler structures on the cones $\mathbb{S}^3 \times \mathbb{R}_+$ and $\text{AdS}^{2+1} \times \mathbb{R}_+$.

Proposition

Let (M, η, g) be a Sasakian manifold and $\mathbf{K}$ a Killing vector field. Then, $\iota_{\mathbf{K}}\eta$ is a first integral of $\mathcal{R}$.

From this one can obtain contact Lie systems of Liouville type on $\mathbb{S}^3$ and AdS^{2+1} .

- $\mathbb{S}^3 = \mathbb{S}^3_{[+],+,+}$ and $\text{AdS}^{2+1} = \mathbb{S}^3_{[+],-,+}$ are *Sasakian manifolds*. This is,

$$\omega = d(s^2 \eta_{\kappa}), \quad ds \otimes ds + s^2 g_{\kappa}$$

define Kähler structures on the cones $\mathbb{S}^3 \times \mathbb{R}_+$ and $\text{AdS}^{2+1} \times \mathbb{R}_+$.

Proposition

Let (M, η, g) be a Sasakian manifold and $\mathbf{K}$ a Killing vector field. Then, $\iota_{\mathbf{K}}\eta$ is a first integral of $\mathcal{R}$.

From this one can obtain contact Lie systems of Liouville type on $\mathbb{S}^3$ and AdS^{2+1} .

Proposition

The contact Lie systems of Liouville type contained within the $\mathfrak{sp}(4, \mathbb{R})$ -contact Lie system h_κ on $\mathbb{S}^3$ and AdS^{2+1} produce, through the Hopf fibrations $\mathbb{S}^3 \rightarrow \mathbb{S}^2$ and $\text{AdS}^{2+1} \rightarrow \mathbb{CH}^1$, the $\mathfrak{so}(3)$ and $I_4 \simeq \mathfrak{sl}(2, \mathbb{R})$ Lie–Hamilton classes on $\mathbb{S}^2$ and $\mathbb{CH}^1$, respectively.

Outline

- 1 Lie systems
Compatible geometric structures
- 2 3D CK spaces
- 3 Contact Lie systems on 3D CK spaces
- 4 Conclusions and open problems
- 5 References

- Invariant contact structures for the Cayley–Klein family.



D. V. Alekseevskii.

Contact homogeneous spaces.

Funct. Anal. Its Appl. **24** (1990) 324–325.

- Deformations of $\mathfrak{so}(3, 1)$ -contact Lie systems on $S^3_{[\kappa_1], \kappa_2, \kappa_3}$ from the nonstandard quantum Lorentz algebra.



F. J. Herranz.

New time-type and space-type non-standard quantum algebras and discrete symmetries.

J. Phys. A: Math. Gen **33** (2000) 8217.

- Invariant contact structures for the Cayley–Klein family.



D. V. Alekseevskii.

Contact homogeneous spaces.

Funct. Anal. Its Appl. **24** (1990) 324–325.

- Deformations of $\mathfrak{so}(3, 1)$ -contact Lie systems on $\mathbb{S}^3_{[\kappa_1], \kappa_2, \kappa_3}$ from the nonstandard quantum Lorentz algebra.



F. J. Herranz.

New time-type and space-type non-standard quantum algebras and discrete symmetries.

J. Phys. A: Math. Gen **33** (2000) 8217.

Outline

- 1 Lie systems
Compatible geometric structures
- 2 3D CK spaces
- 3 Contact Lie systems on 3D CK spaces
- 4 Conclusions and open problems
- 5 References

References



J. de Lucas and C. Sardón.

A Guide to Lie Systems with Compatible Geometric Structures.

(Singapore: World Scientific) 2020.

doi:10.1142/q0208



J. de Lucas and X. Rivas.

Contact Lie systems: theory and applications.

J. Phys. A: Math. Theor. **56** (2023) 335203.

doi:10.1088/1751-8121/ace0e7



R. Campoamor-Stursberg, O. Carballal and F. J. Herranz.
A representation-theoretical approach to higher-dimensional
Lie–Hamilton systems: The symplectic Lie algebra $\mathfrak{sp}(4, \mathbb{R})$.
Commun. Nonlinear Sci. Numer. Simul. **141** (2025) 108452.
[doi:10.1016/j.cnsns.2024.108452](https://doi.org/10.1016/j.cnsns.2024.108452)

Thank you for your kind attention!

Gracias :)

✉ oscarbal@ucm.es

Thank you for your kind attention!

Gracias :)

✉ oscarbal@ucm.es