Curvature-dependent reductions of contact Lie systems on curved spaces

XIX$^{\text{th}}$ Young Researchers Workshop in Geometry, Dynamics and Field Theory

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Joint work with R. Campoamor-Stursberg (UCM) and F. J. Herranz (UBU)

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Outline

Consider the first-order *Riccati equation* on $\mathbb{R}$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_0(t) + a_1(t)x + a_2(t)x^2, \qquad a_\alpha \in C^\infty(\mathbb{R}).$$

- There is no general method for obtaining its general solution!
- Lie, Vessiot and Guldberg (among others) noticed that the general solution $x(t)$ can be expressed as

$$x(t) = \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) + kx_{(2)}(t)(x_{(1)}(t) - x_{(3)}(t))}{x_{(3)}(t) - x_{(2)}(t) + k(x_{(1)}(t) - x_{(3)}(t))}$$

  with $x_{(i)}(t)$ a particular solution and $k \in \mathbb{R} \cup \{\infty\}$.

Such expression is known as a *superposition rule*.

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$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \psi^i(t,x), \qquad x \in M, \qquad 1 \leq i \leq n := \dim M.$$

Its solutions are the integral curves of the t-dependent vector field

$$\mathbf{X} : \mathbb{R} \times M \to \mathrm{T}M$$

locally given by $\mathbf{X}(t,x) = \psi^i(t,x)\frac{\partial}{\partial x^i}$.

$$\mathbf{X} \rightsquigarrow \{\mathbf{X}_t\}_{t\in\mathbb{R}} \subset \mathfrak{X}(M) \text{ such that } \mathbf{X}_t(x) := \mathbf{X}(t,x).$$

$$\left\{ \begin{array}{c} \text{1st order ODEs systems} \\ \text{in normal form} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} t\text{-dependent} \\ \text{vector fields} \end{array} \right\}$$

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$$\Psi : M^s \times M \to M$$

such that the general solution $\mathbf{x}(t)$ of $\mathbf{X}$ can be expressed as

$$\mathbf{x}(t) = \Psi(\mathbf{x}_{(1)}(t), \ldots, \mathbf{x}_{(s)}(t); k),$$

where $\mathbf{x}_{(1)}(t), \ldots, \mathbf{x}_{(s)}(t)$ is a generic family of particular solutions and $k \in M$ is a point related to the initial conditions.

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$\mathbf{X}$ is a Lie system if and only if it admits the decomposition

$$\mathbf{X}(t,\mathbf{x}) = \sum_{\alpha=1}^{r} b_\alpha(t)\mathbf{X}_\alpha$$

for some functions $b_1,\ldots,b_r \in C^\infty(\mathbb{R})$ and vector fields $\mathbf{X}_1,\ldots,\mathbf{X}_r$ on M spanning an r-dimensional real Lie algebra V^X.

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$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = x\frac{\partial}{\partial x}, \quad \mathbf{X}_3 = x^2\frac{\partial}{\partial x}$$

have commutation rules

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3,$$

and therefore span a VG Lie algebra $V^X \simeq \mathfrak{sl}(2, \mathbb{R})$.

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$\dim M = 2n,\ \mathrm{d}\omega = 0,\ \omega^n \neq 0$	$\dim M = 2n+1,\ \eta\wedge(\mathrm{d}\eta)^n \neq 0$
	$\iota_{\mathcal{R}}\eta = 1,\ \iota_{\mathcal{R}}\mathrm{d}\eta = 0$
$\iota_{\mathbf{X}_f}\omega = \mathrm{d}f$	$\iota_{\mathbf{X}_f}\eta = -f,\ \iota_{\mathbf{X}_f}\mathrm{d}\eta = \mathrm{d}f - (\mathcal{R}f)\eta$
$\{f,g\}_\omega = \omega(\mathbf{X}_f,\mathbf{X}_g)$	$\{f,g\}_\eta = \mathbf{X}_f g + g\mathcal{R}f$

Definition

- A Lie system $\mathbf{X}$ on (M,ω) is a *Lie–Hamilton system* if V^X is formed by Hamiltonian vector fields relative to ω.
- A Lie system $\mathbf{X}$ on (M,η) is a *contact Lie system* if V^X is formed by contact Hamiltonian vector fields relative to η. If $\mathcal{R}$ is a Lie symmetry of V^X, then $\mathbf{X}$ is called a contact Lie system *of Liouville type*.

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The integral curves of $\mathbf{X}$ are the solutions of the Hamilton equations of a *t-dependent Hamiltonian*

$$h : \mathbb{R} \times M \to \mathbb{R}, \qquad h(t,x) := h_t(x),$$

with $\mathbf{X}_t$ the Hamiltonian vector field of h_t.

Definition (Lie–Hamilton algebra of $\mathbf{X}$)

- $\mathcal{H}_\omega := \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\omega)$.
- $\mathcal{H}_\eta := \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\eta)$.

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- $\mathrm{SO}_{\kappa_1,\kappa_2,\kappa_3}(4)$ the Lie group of isometries of

$$\mathbb{I}_{\boldsymbol{\kappa}} := \mathrm{diag}(1, \kappa_1, \kappa_1\kappa_2, \kappa_1\kappa_2\kappa_3) \text{ on } \mathbb{R}^4, \text{ with } \kappa_i \in \{0, \pm 1\}\,.$$

- H_0 the isotropy subgroup of the point $O := (1, 0, 0, 0)$.

Definition (3D Cayley–Klein space)

$$\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3} := \mathrm{SO}_{\kappa_1,\kappa_2,\kappa_3}(4)/H_0.$$

- $g_{\boldsymbol{\kappa}}$ the projection of metric on $\mathrm{SO}_{\kappa_1,\kappa_2,\kappa_3}(4)$ obtained by right translation of $\mathbb{I}_{\boldsymbol{\kappa}}$.

κ_1 is a (sectional) *curvature* and κ_2, κ_3 determine the *signature* through $\mathrm{diag}(1, \kappa_2, \kappa_2\kappa_3)$.

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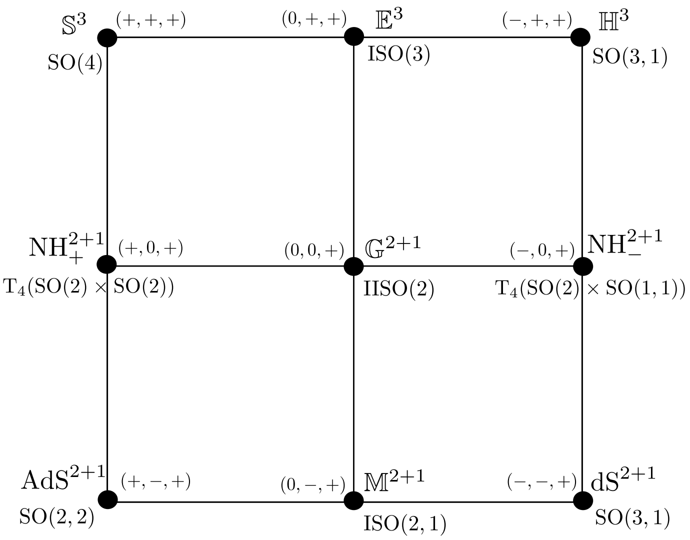
Kinematical spaces $\mathbb{S}^3_{[\kappa_1],\kappa_2,+}$ *à la Bacry & Lévy-Leblond*



- $\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3}$ is diffeomorphic to the connected component of

$$\left\{(x^0)^2+\kappa_1(x^1)^2+\kappa_1\kappa_2(x^2)^2+\kappa_1\kappa_2\kappa_3(x^3)^2=1\right\}=\mathbb{I}_{\boldsymbol{\kappa}}^{-1}(1)$$

containing $O=(1,0,0,0)$.

- Exact symplectic manifold $(\mathbb{R}^4_0:=\mathbb{R}^4-\{0\},\omega=\mathrm{d}x^0\wedge\mathrm{d}x^1+\mathrm{d}x^2\wedge\mathrm{d}x^3)$ and Liouville vector field

$$\boldsymbol{\Delta}:=\frac{1}{2}\left(x^0\frac{\partial}{\partial x^0}+x^1\frac{\partial}{\partial x^1}+x^2\frac{\partial}{\partial x^2}+x^3\frac{\partial}{\partial x^3}\right)\rightsquigarrow\omega=\mathrm{d}\iota_{\boldsymbol{\Delta}}\omega.$$

- $\boldsymbol{\Delta}$ is transversal to $i:\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3}\hookrightarrow\mathbb{R}^4_0$, so

$$\eta_{\boldsymbol{\kappa}}:=i^*\iota_{\boldsymbol{\Delta}}\omega$$

is a contact form on $\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3}$.

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Theorem

There exists an invariant $\mathrm{SO}_{\kappa_1,\kappa_2,\kappa_3}(4)$*-connection* ∇ *on* $\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3}$ *such that* $\nabla_{\mathcal{R}_{\boldsymbol{\kappa}}}\mathcal{R}_{\boldsymbol{\kappa}} = 0$.

The integral curves of $\mathcal{R}_{\boldsymbol{\kappa}}$ define a one-dimensional foliation $\mathcal{F}_{\boldsymbol{\kappa}}$ of $\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3}$ by geodesics.

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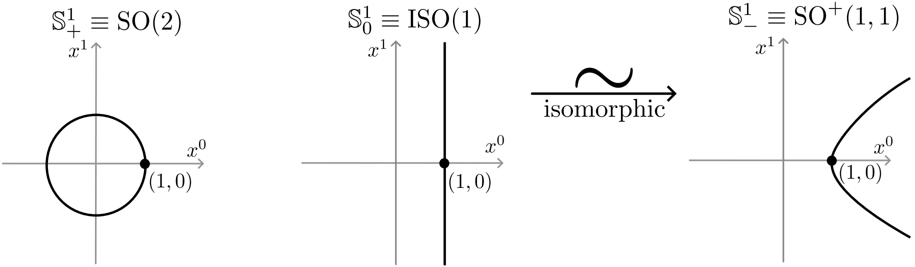

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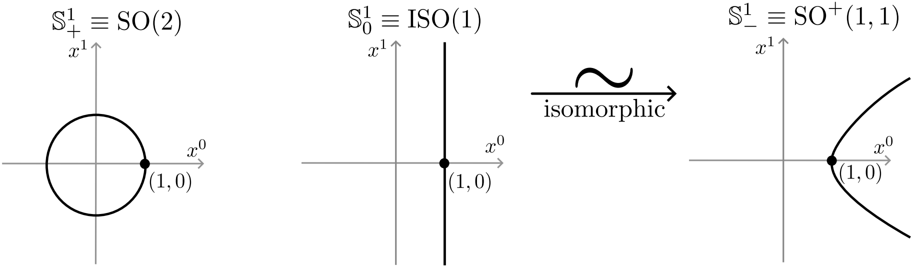

Theorem

There exists an invariant $\mathrm{SO}_{\kappa_1,\kappa_2,\kappa_3}(4)$*-connection* ∇ *on* $\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3}$ *such that* $\nabla_{\mathcal{R}_{\boldsymbol{\kappa}}}\mathcal{R}_{\boldsymbol{\kappa}} = 0$.

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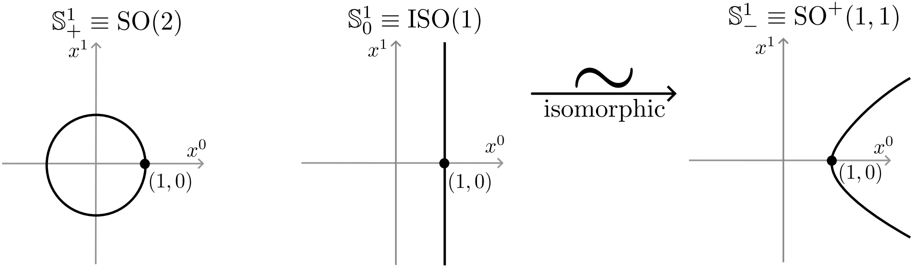

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The foliation $\mathcal{F}_{\boldsymbol{\kappa}}$ on $\mathbb{S}^3_{[\kappa_1],\kappa_2,+}$ is regular when $(\kappa_1,\kappa_2)\neq(-1,-1)$, inducing an $\mathbb{S}^1_{[\kappa_1]}$-principal bundle $\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3}\to\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3}/\mathcal{R}_{\boldsymbol{\kappa}}$.

From this one recovers the Hopf fibrations

$$\mathbb{S}^3\to\mathbb{S}^2,\qquad \mathrm{AdS}^{2+1}\to\mathbb{CH}^1.$$

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Outline

- The Hamiltonian functions $h_\alpha \in C^\infty(\mathbb{R}^4_0)$ given by

$$h_1 = x^0x^1, \quad h_2 = x^0x^3, \quad h_3 = x^1x^2, \quad h_4 = x^2x^3, \quad h_5 = \frac{1}{2}(x^0)^2,$$
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  span a Lie–Hamilton algebra $\mathcal{H}_\omega \simeq \mathfrak{sp}(4,\mathbb{R})$ with respect to $\{\cdot,\cdot\}_\omega$.
- Lie–Hamilton system on $(\mathbb{R}^4_0, \omega)$ with t-dependent Hamiltonian $h := \sum_{\alpha=1}^{10} b_\alpha(t) h_\alpha$.
- The contact Hamiltonian functions $h_{\kappa,\alpha} := i^* h_\alpha \in C^\infty(\mathbb{S}^3_{[\kappa_1],\kappa_2,\kappa_3})$ span a Lie–Hamilton algebra $\mathcal{H}_{\eta_\kappa} \simeq \mathfrak{sp}(4,\mathbb{R})$ with respect to $\{\cdot,\cdot\}_{\eta_\kappa}$.
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$$\mathrm{d}\left(s^2\eta_{\boldsymbol{\kappa}}\right), \qquad \mathrm{d}s \otimes \mathrm{d}s + s^2 g_{\boldsymbol{\kappa}}$$

define Kähler structures on the cones $\mathbb{S}^3 \times \mathbb{R}_+$ and $\mathrm{AdS}^{2+1} \times \mathbb{R}_+$.

Proposition

Let (M, η, g) be a Sasakian manifold and $\mathbf{K}$ a Killing vector field. Then, $\iota_{\mathbf{K}}\eta$ is a first integral of $\mathcal{R}$.

From this one can obtain contact Lie systems of Liouville type on $\mathbb{S}^3$ and AdS^{2+1}.

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Proposition

The contact Lie systems of Liouville type contained within the $\mathfrak{sp}(4,\mathbb{R})$*-contact Lie system* $h_{\boldsymbol{\kappa}}$ *on* $\mathbb{S}^3$ *and* AdS^{2+1} *produce, through the Hopf fibrations* $\mathbb{S}^3 \to \mathbb{S}^2$ *and* $\mathrm{AdS}^{2+1} \to \mathbb{CH}^1$*, the* $\mathfrak{so}(3)$ *and* $\mathrm{I}_4 \simeq \mathfrak{sl}(2,\mathbb{R})$ *Lie–Hamilton classes on* $\mathbb{S}^2$ *and* $\mathbb{CH}^1$*, respectively.*

Outline

- Invariant contact structures for the Cayley–Klein family.

D. V. Alekseevskii.
Contact homogeneous spaces.
Funct. Anal. Its Appl. **24** (1990) 324–325.

- Deformations of contact Lie systems through quantum groups.

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C_z^\infty(\mathfrak{g}^*) \simeq C^\infty(\mathfrak{g}^*)\widehat{\otimes}\mathbb{R}[[z]] & \xrightarrow{D_z} & (C^\infty(M), \{\cdot,\cdot\}_\eta) \\
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Outline

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Thank you for your kind attention!

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