Mixed superposition rules for Lie systems: compatible geometric structures and applications

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Outline

The **Riccati equation** on $\mathbb{R}$:

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = b_0(t) + b_1(t)x + b_2(t)x^2.}$$

- In general, it cannot be solved by quadratures!
- If $x_{(1)}(t), \ldots, x_{(4)}(t)$ are four *different* particular solutions, their cross ratio is constant:

$$\frac{(x_{(1)} - x_{(4)})(x_{(2)} - x_{(3)})}{(x_{(1)} - x_{(2)})(x_{(3)} - x_{(4)})} = k.$$

- And even more important,

$$x_{(1)}(t) = \frac{x_{(4)}(t)(x_{(2)}(t) - x_{(3)}(t)) + kx_{(2)}(t)(x_{(4)}(t) - x_{(3)}(t))}{(x_{(2)}(t) - x_{(3)}(t)) + k(x_{(4)}(t) - x_{(3)}(t))}$$

is the **generic solution**.

Geometric fundamentals

- **Time-dependent vector field**: $X : \mathbb{R} \times M \to \mathrm{T}M$ such that

$$\begin{tikzcd} \mathbb{R} \times M \arrow[r, "X"] \arrow[dr] & \mathrm{T}M \arrow[d] \\ & M \end{tikzcd} \iff \{X_t := X(t, \cdot)\}_{t \in \mathbb{R}} \subset \mathfrak{X}(M)$$

- **Associated system of ODEs**: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = X(t, x)$
- **Superposition rule**: $\Phi : M^\ell \times M \dashrightarrow M$ such that the generic solution $x(t)$ can be cast in the form

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(\ell)}(t); k)$$

where $x_{(1)}(t), \ldots, x_{(\ell)}(t)$ is a generic family of particular solutions and $k \in M$ is a point related to the initial conditions.

Lie systems

Definition (Lie system)

A **Lie system** on M is a time-dependent vector field $X : \mathbb{R} \times M \to TM$ taking values in a *finite-dimensional Lie algebra* $V \subset \mathfrak{X}(M)$ $(\{X_t\}_{t\in\mathbb{R}} \subset V)$. The Lie algebra V is a so-called **Vessiot–Guldberg Lie algebra** associated with X.

Theorem (Lie)

X admits a superposition rule if and only if it is a Lie system.

Example (Riccati equation)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b_0(t) + b_1 x + b_2(t)x^2 \rightsquigarrow X = b_0(t) \underbrace{\frac{\partial}{\partial x}}_{X_1} + b_1(t) \underbrace{x\frac{\partial}{\partial x}}_{X_2} + b_2(t) \underbrace{x^2\frac{\partial}{\partial x}}_{X_3},$$

$$V \simeq \mathfrak{sl}(2,\mathbb{R}) : \qquad [X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.$$

Outline

Question ($\star$)

Under which conditions the generic solution of a system X can be expressed in terms of a generic family of particular solutions of (possibly different) systems $X_{(1)}, \ldots, X_{(\ell)}$ and some constants related to the initial conditions?

Definition (Mixed superposition rule)

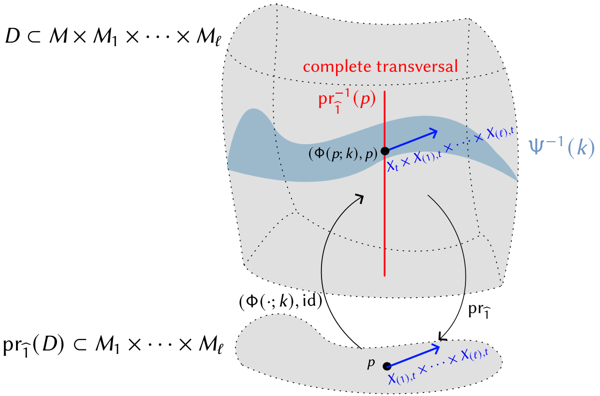
A **mixed superposition rule** for a system X on M in terms of systems $X_{(i)}$ on M_i ($1 \leq i \leq \ell$) is a map $\Phi : M_1 \times \cdots \times M_\ell \times M \dashrightarrow M$ through which ($\star$) holds.

Theorem (Grabowski, de Lucas)

X admits a mixed superposition rule if and only if it is a Lie system.

Example

- Inhomogeneous linear systems.
- Riccati hierarchy (in terms of homogeneous linear ODEs).



$$\left\{ \begin{array}{c} \text{mixed superposition rules} \\ \Phi : M \times M_1 \times \cdots \times M_\ell \dashrightarrow M \end{array} \right\} \leftrightharpoons \left\{ \begin{array}{c} \text{local inverses of} \\ (\mathrm{pr}_{\hat{1}}, \Psi) \end{array} \right\},$$

where $\Psi : M \times M_1 \times \cdots \times M_\ell \dashrightarrow M$ is a submersion such that

$$\mathrm{T}\Psi(X_t \times X_{(1),t} \times \cdots X_{(\ell),t}) = 0, \qquad t \in \mathbb{R}.$$

Imprimitive VG Lie algebras

- $V \subset \mathfrak{X}(M)$ is **imprimitive** if there exists a regular and integrable distribution $\mathcal{D} \subsetneq \mathrm{T}M$ such that $[V, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$.
- Provided that $M/\mathcal{D}$ is a manifold and $\pi : M \to M/\mathcal{D}$ is a submersion,

$$0 \longrightarrow V \cap \Gamma(\mathcal{D}) \longrightarrow V \xrightarrow{\pi_*} \pi_* V \longrightarrow 0$$

Let now X be a Lie system such that V is a VG Lie algebra of it.

Theorem (C., Campoamor-Stursberg, de Lucas, Herranz)

(1) *If $V \cap \Gamma(\mathcal{D}) = 0$, then X admits a mixed superposition rule depending uniquely on particular solutions of $\pi_* X$.*

(2) *If $V \cap \Gamma(\mathcal{D}) \neq 0$, then X admits a mixed superposition rule depending on particular solutions of X and $\pi_* X$.*

Semi-direct sums

Let X be a Lie system possessing a VG Lie algebra $V = V_1 \ltimes V_2$. The **part of X taking values in V_1** is the (unique) t-dependent vector field Y such that $Y_t \in V_1$ and $X_t - Y_t \in V_2$ for all $t \in \mathbb{R}$.

Proposition (C., Campoamor-Stursberg, de Lucas, Herranz)

X admits a mixed superposition rule depending on s particular solutions of X and k particular solutions of Y, where $s, k \geq 1$ are such that the $V_1^{[k]}$ and $V_2^{[s]}$ possess a modular basis.

Outline

Dirac geometry

Consider $\mathbb{T}M := \mathrm{T}M \oplus_M \mathrm{T}^*M$ equipped with

- the symmetric pairing $\langle X_x + \alpha_x, Y_x + \beta_x \rangle_+ := \frac{1}{2}(\alpha_x(Y_x) + \beta_x(X_x))$, and
- the Dorfman bracket $[\![\cdot,\cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ defined by

$$[\![X + \alpha, Y + \beta]\!] := [X, Y] + \mathcal{L}_X\beta - \iota_Y \mathrm{d}\alpha.$$

Definition (Dirac manifold)

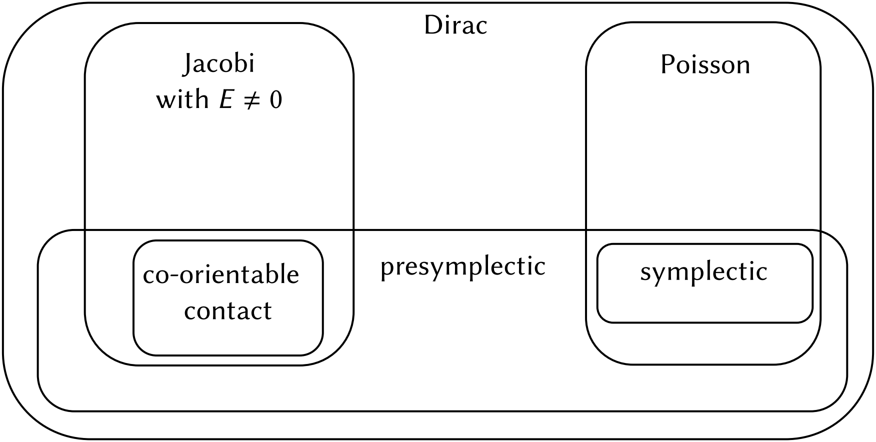
A **Dirac manifold** is a pair $(M, \mathbb{L})$, where $\mathbb{L} \subset \mathbb{T}M$ is a vector subbundle such that:

- $\mathbb{L}$ is Lagrangian with respect to $\langle \cdot, \cdot \rangle_+$, and
- $\mathbb{L}$ is involutive with respect to $[\![\cdot,\cdot]\!]$.

Example

$\omega \in \Omega^2(M)$ and $\Pi \in \Gamma(\wedge^2 \mathrm{T}M)$ determine morphisms $\omega^\flat : \mathrm{T}M \to \mathrm{T}^*M$ and $\Pi^\sharp : \mathrm{T}^*M \to \mathrm{T}M$. Their graphs $\mathbb{L}^\omega$ and $\mathbb{L}^\Pi$ are Dirac structures if and only if $\mathrm{d}\omega = 0$ and $[\Pi, \Pi] = 0$, respectively.

Some relations (relative to standard inclusions)



Dynamics on Dirac manifolds

- $f \in C^\infty(M)$ is **$\mathbb{L}$-admissible** if there exists $X_f \in \mathfrak{X}(M)$, a so-called **Hamiltonian vector field** associated with f, such that $X_f + \mathrm{d}f \in \Gamma(\mathbb{L})$.
- $\mathrm{Adm}(M, \mathbb{L}) := \{f \in C^\infty(M) : f \text{ is } \mathbb{L}\text{-admissible}\}$ is a **Poisson algebra**:

$$\{f, g\}_{\mathbb{L}} := X_f g \qquad \text{does not depend on } X_f!$$

- If $f_1, \ldots, f_r \in \mathrm{Adm}(M, \mathbb{L})$ and $F \in C^\infty(\mathbb{R}^r) \Rightarrow F(f_1, \ldots, f_r) \in \mathrm{Adm}(M, \mathbb{L})$.
- $\phi : \mathfrak{g} \to \mathrm{Adm}(M, \mathbb{L})$ morphism of Lie algebras. Its associated **moment map $\mathbf{J}_\phi$** $: M \to \mathfrak{g}^*$, defined by

$$\langle \mathbf{J}_\phi(x), v \rangle := \phi(v)(x),$$

is such that its pull-back, $\mathbf{J}_\phi^* : C^\infty(\mathfrak{g}^*) \to \mathrm{Adm}(M, \mathbb{L})$, is a morphism of Poisson algebras.

Dirac–Lie systems

Definition

A **Dirac–Lie system** is a triple $(M, \mathbb{L}, X)$, where $(M, \mathbb{L})$ is a Dirac manifold and X is a Lie system on M possessing a VG Lie algebra V consisting of Hamiltonian vector fields relative to $\mathbb{L}$.

- There exists a finite-dimensional Lie algebra $\mathcal{H} \subset \mathrm{Adm}(M, \mathbb{L})$ such that

$$V = \left\{ X_f : f \in \mathcal{H} \right\}.$$

- Let $\mathfrak{g} \simeq \mathcal{H}$ and $\phi : \mathfrak{g} \longrightarrow \mathrm{Adm}(M, \mathbb{L})$ a morphism of Lie algebras such that $\phi(\mathfrak{g}) = \mathcal{H}$.

$$\mathscr{C} \in C^\infty(\mathfrak{g}^*) \text{ Casimir function} \Rightarrow \mathbf{J}_\phi^* \mathscr{C} \text{ first integral of } X.$$

Remark

Mixed superposition rules can be constructed algorithmically by means of of well-behaved Poisson coalgebras, moment maps, comodule Poisson algebras and coactions.

A time-dependent thermodynamical system

Thermodynamical phase space $(\mathbb{R}^5, \eta_{\mathrm{Gibbs}})$ with:

- extensive variables (U, S, v);
- intensive variables (T, P), and
- the Gibbs one-form

$$\eta_{\mathrm{Gibbs}} := \mathrm{d}U - T\mathrm{d}S + P\mathrm{d}v, \qquad (\eta_{\mathrm{Gibbs}} \wedge (\mathrm{d}\eta_{\mathrm{Gibbs}})^2 \neq 0).$$

Lie system $X := \sum_{i=1}^{6} b_i(t) X_i$:

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}t} &= -\big(b_1(t) - b_6(t)\big)U + b_3(t)v + b_5(t)S,\\
\frac{\mathrm{d}S}{\mathrm{d}t} &= b_6(t)S,\\
\frac{\mathrm{d}v}{\mathrm{d}t} &= b_2(t)U + \big(b_1(t) + b_6(t)\big)v + b_4(t)S,\\
\frac{\mathrm{d}T}{\mathrm{d}t} &= -b_1(t)T + b_2(t)TP + b_4(t)P + b_5(t),\\
\frac{\mathrm{d}P}{\mathrm{d}t} &= b_2(t)P^2 - 2b_1(t)P - b_3(t),
\end{aligned}$$

where the vector fields

$$X_1 := -U\frac{\partial}{\partial U} + v\frac{\partial}{\partial v} - T\frac{\partial}{\partial T} - 2P\frac{\partial}{\partial P}, \qquad X_2 := U\frac{\partial}{\partial v} + TP\frac{\partial}{\partial T} + P^2\frac{\partial}{\partial P},$$

$$X_3 := v\frac{\partial}{\partial U} - \frac{\partial}{\partial P}, \qquad X_4 := S\frac{\partial}{\partial v} + P\frac{\partial}{\partial T},$$

$$X_5 := S\frac{\partial}{\partial U} + \frac{\partial}{\partial T}, \qquad X_6 := U\frac{\partial}{\partial U} + S\frac{\partial}{\partial S} + v\frac{\partial}{\partial v}.$$

span a VG Lie algebra $V \simeq \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^3$.

- $V' := \langle X_1, X_3, X_4, X_5, X_6 \rangle \simeq \mathfrak{b}_2 \ltimes \mathbb{R}^3$, and

$$X_1 \wedge X_3 \wedge X_4 \wedge X_4 \wedge X_6 = -S^2 \underbrace{(U + Pv - ST)}_{\text{Gibbs free energy}} \frac{\partial}{\partial U} \wedge \frac{\partial}{\partial S} \wedge \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial T} \wedge \frac{\partial}{\partial P}.$$

- Restrict to $M := \{S(U + Pv - ST) \neq 0\} \subset \mathbb{R}^5$.

- $\mathrm{Sym}(V') \simeq \mathfrak{b}_2 \ltimes \mathbb{R}^3$ yields a dual co-frame $\{\alpha_1, \ldots, \alpha_5\}$ on M

$$\alpha_1 := -\frac{1}{U + Pv - ST}\left(\mathrm{d}U - \frac{U + Pv}{S}\mathrm{d}S + P\mathrm{d}v - S\mathrm{d}T + v\mathrm{d}P\right),$$

$$\alpha_2 := -\left(\frac{S}{U + Pv - ST}\right)^2 \mathrm{d}P, \qquad \alpha_3 := \left(\frac{U + Pv - ST}{S^2}\right)\left(\frac{v}{S}\mathrm{d}S - \mathrm{d}v\right)$$

$$\alpha_4 := \frac{1}{U + Pv - ST}(-S\mathrm{d}T + v\mathrm{d}P), \qquad \alpha_5 := -\frac{1}{S}\mathrm{d}S,$$

satisfying the Maurer–Cartan equations of $\mathfrak{b}_2 \ltimes \mathbb{R}^3$:

$$\mathrm{d}\alpha_2 = 2\alpha_1 \wedge \alpha_2, \quad \mathrm{d}\alpha_3 = -\alpha_1 \wedge \alpha_3, \quad \mathrm{d}\alpha_4 = \alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3, \quad \mathrm{d}\alpha_1 = \mathrm{d}\alpha_5 = 0.$$

- Remarkably,

$$\eta := -\alpha_1 - \alpha_4 - \alpha_5 = \frac{1}{U + Pv - ST}\,\eta_{\mathrm{Gibbs}} = \frac{1}{U + Pv - ST}(\mathrm{d}U - T\mathrm{d}S + P\mathrm{d}v)$$

is a contact form on M such that X is a **Dirac–Lie system relative to the presymplectic form** $\mathrm{d}\eta$.

Mixed superposition rule for the intensive system

$$T_{(1)}(t) = \frac{1}{S_{(1)}(t)\Theta(t)}\Big[k_1 k_3^2\big[U_{(1)}(t) + P_{(2)}(t)v_{(1)}(t)\big] - k_3 S_{(1)}(t)\big[U_{(2)}(t) + P_{(2)}(t)v_{(2)}(t)\big]$$
$$+ k_2\big[U_{(2)}(t) + P_{(2)}(t)v_{(2)}(t) - k_3 T_{(2)}(t)\big]\big[U_{(2)}(t)v_{(1)}(t) - U_{(1)}(t)v_{(2)}(t)\big]\Big],$$

$$P_{(1)}(t) = \frac{1}{\Theta(t)}\Big[k_1 k_3^2 P_{(2)}(t) + k_2 U_{(2)}(t)\big[U_{(2)}(t) + P_{(2)}(t)v_{(2)}(t) - k_3 T_{(2)}(t)\big]\Big],$$

where

$$\Theta(t) := k_1 k_3^2 - k_2 v_{(2)}(t)\big[U_{(2)}(t) + P_{(2)}(t)v_{(2)}(t) - k_3 T_{(2)}(t)\big],$$

in terms of three constants $k_1, k_2, k_3 \in \mathbb{R}$ and

- one particular solution $\big(U_{(1)}(t), S_{(1)}(t), v_{(1)}(t)\big)$ of the extensive system;
- one particular solution $\big(U_{(2)}, S_{(2)}(t) = k_3, v_{(2)}(t)\big)$ extensive system with constant entropy, and
- one particular solution $\big(T_{(2)}(t), P_{(2)}(t)\big)$ of the intensive system associated with the Levi factor of $V \simeq \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^3$:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -b_1(t)T + b_2(t)TP, \quad \frac{\mathrm{d}P}{\mathrm{d}t} = b_2(t)P^2 - 2b_1(t)P - b_3(t).$$

Outline

Conclusions and further work

- The general solution of the Liouville equation $u_{xy} = \mathrm{e}^u$ can be expressed as

$$u(x, y) := \log \frac{2\dot{f}(x)\dot{g}(y)}{(f(x) + g(y))^2}.$$

- Darboux integrable PDEs admit these "mixed superposition rules".

Some ideas:

- Characterise PDEs admitting "mixed superposition rules", generalising Lie Theorem and Darboux integrability.

In terms of...

- morphisms of Pfaffian fibrations / underlying relative algebroids (Smilde and Fernandes, '25).
- pseudogroups of symmetries and associated Lie algebra sheaves.

Outline

References

[1] R. Campoamor-Stursberg, O. Carballal, J. de Lucas and F. J. Herranz.
Mixed superposition rules for Lie systems and compatible geometric structures.
Preprint arXiv:2511.01063 [math-ph].

[2] J. Grabowski and J. de Lucas.
Mixed superposition rules and the Riccati hierarchy.
J. Differential Equations **254** (2013) 179–198.

[3] J. F. Cariñena, J. Grabowski, J. de Lucas and C. Sardón.
Dirac–Lie systems and Schwarzian equations.
J. Differential Equations **257** (2014) 2303–2340.

[4] I. Anderson, M. Fels and P. Vassiliou.
Superposition formulas for exterior differential systems.
Adv. Math. **221** (2009) 1910–1963.

Thank you for your kind attention!

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